Lecture Notes on Infinite Series

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Definition 1 *An Infinite Series is an expression of the form*
$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$

Where $a_1, a_2, a_3, ..., a_n, ...$ are called the terms of the series. If we let S_n be the sum of the first n terms of the series then we have the following:
$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
.
.
.
$S_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{n=1}^{n} a_n$
We'll call S_n the n^{th} partial sum of the series.
The partial sums form $\{S_n\}_{n=1}^{+\infty}$ - the sequence of partial sums.

Definition 2 *Let $\{S_n\}$ be a sequence of partial sums of $\sum_{n=1}^{\infty} a_n$.*
If $\{S_n\}$ converges to a limit S, then the series also converges and S is called the sum of the series.

$$S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges then the series is said to diverge.
A divergent series has no sum.

Example 1 *Determine if the series $1 - 1 + 1 - 1 + 1 - 1 + ...$ converges or diverges.*

Now, $S_1 = 1, S_2 = 1 - 1 = 0, S_3 = 1, S_4 = 0,$ *e.t.c*
$1, 0, 1, 0, 1, 0, ...$ *is the sequence of partial sums.*
This sequence is divergent $\Rightarrow$ *the given series is divergent.*

We'll now look at a class of series called a geometric series.

Definition 3 *A geometric series is a series of the form* $a + ar + ar^2 + ar^3 + ... + ar^n + ...$ *Where* $a \neq 0$ *and* r *is a real number called the ratio of the series.*

Theorem 1 *A geometric series converges if* $|r| < 1$ *and diverges if* $|r| \geq 1$. *When the series converges the sum is* $\frac{a}{1-r}$

Example 2 $5 + \frac{5}{4} + \frac{5}{4^2} + \frac{5}{4^3} + \frac{5}{4^4} + ... + \frac{5}{4^{k-1}} + ...$
is a geometric series with $a = 5,$ *and* $r = \frac{1}{4}$. $\Rightarrow$ *the series converges with sum* $= \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$

Example 3 *Determine if* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ *converges or diverges. If it converges find the sum.*
Well I'll leave this one for you. Just identify it as a geometric series and do what's needed.

Example 4 *Determine if the series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ *converges or diverges. If it converges find its sum.*
Here, $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{n(n+1)}$
Using partial fractions we see that
$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$
This implies that $S_n = \displaystyle\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + ... + \frac{1}{n} - \frac{1}{n+1}$

$$= 1 - \frac{1}{n+1}$$

So $S_n = 1 - \frac{1}{n+1}$ *and* $\displaystyle\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right) = 1$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$$

Which means that the series converges with sum 1.
The series in Example 4 is an example of what we call a Telescoping series.

Example 5 *Determine if* $\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$ *converges or diverges. If it converges find the sum.*
This is another one that I would like you to try for me.

We now come to an important theorem that allows us to quickly decide if a series diverges or not.

Theorem 2 (Divergence Test) *If* $\lim_{k\to\infty} a_k \neq 0$ *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Example 6 $\sum_{k=1}^{\infty} \frac{k}{k+1}$ *diverges since*

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \neq 0$$

Theorem 3 *(Properties of Infinite Series)*

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
2. $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$
3. *Convergence and Divergence are unaffected by deleting a finite number of terms from the beginning of a series.*

From (1) we see that if a series is convergent then a scalar times that series is also convergent. Similarly, if a series diverges then a scalar times that series also diverges.
From (2) it is obvious that the sum or difference of 2 convergent series also converges.

Example 7 *Find the sum of the series* $\sum_{k=1}^{\infty}\left(\frac{3}{4^k}-\frac{2}{5^{k-1}}\right)$.

From the above theorem $\sum_{k=1}^{\infty}\left(\frac{3}{4^k}-\frac{2}{5^{k-1}}\right)=\sum_{k=1}^{\infty}\frac{3}{4^k}-\sum_{k=1}^{\infty}\frac{2}{5^{k-1}}$

$$=\frac{\frac{3}{4}}{1-\frac{1}{4}}-\frac{2}{1-\frac{1}{5}}$$

$$=1-\frac{5}{2}=-\frac{3}{2}$$

Example 8 *Find the sum of* $\sum_{k=1}^{\infty}\frac{2}{5^k}$

From (1) in Theorem 3, we have $\sum_{k=1}^{\infty}\frac{2}{5^k}=2\sum_{k=1}^{\infty}\frac{1}{5^k}$

Well $\sum_{k=1}^{\infty}\frac{1}{5^k}$ *is a series you already dealt with in Example 3, so you know what to do.*

Example 9 *Determine if* $\sum_{k=10}^{\infty}\frac{k}{k+1}$ *converges or diverges.*

From Example 6 $\sum_{k=1}^{\infty}\frac{k}{k+1}$ *diverges, therefore* $\sum_{k=10}^{\infty}\frac{k}{k+1}$ *also diverges since it is* $\sum_{k=1}^{\infty}\frac{k}{k+1}$ *with the first nine terms taken out and according to (3) from Theorem 3 such a series must also diverge.*

Theorem 4 (Integral Test) *Let* $\sum_{n=1}^{\infty}a_n$ *be a series with positive terms , and let* $f(x)$ *be the function such that* $f(n)=a_n$*. If* f *is decreasing and continuous for* $x\geq 1$ *, then*
$\sum_{n=1}^{\infty}a_n$ *and* $\int_1^{\infty}f(x)\,dx$ *both converge or both diverge.*

Example 10 *Determine if* $\sum_{n=1}^{\infty}\frac{1}{n^2}$ *converges or diverges.*

$$f(x)=\frac{1}{x^2}$$

$$
\begin{aligned}
\int_1^\infty \frac{dx}{x^2} &= \lim_{M\to\infty} \int_1^M \frac{dx}{x^2} \\
&= \lim_{M\to\infty} \left[-\frac{1}{x}\right]_1^M \\
&= \lim_{M\to\infty} \left(1 - \frac{1}{M}\right) = 1
\end{aligned}
$$

We have just shown that the improper integral converges,therefore the series converges.

Example 11 *Show that* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges using the integral test.*
I'll leave this one to you. You just need to set up an improper integral like the one I set up in Example 1. Then show that the integral diverges.

Example 12 *Determine if the series* $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ *converges or diverges.*
Here we'll let $f(x) = \frac{x}{e^{x^2}} = xe^{-x^2}$ *then*

$$f'(x) = e^{-x^2}(1 - 2x^2) \le 0$$

This implies that f *is decreasing for* $x \ge 1$ *and since all the terms of the series are positive we can go ahead and use the integral test.*

$$
\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx &= \lim_{M\to\infty} \int_1^M xe^{-x^2}\,dx \\
&= \lim_{M\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_1^M \\
&= \lim_{M\to\infty} \left[\frac{1}{2e} - \frac{1}{2}e^{-M^2}\right] \\
&= \frac{1}{2e}
\end{aligned}
$$

This implies that the improper integral converges and therefore the series converges.

The Integral Test leads us to the following theorem.

Theorem 5 $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ...,$ *where* $p > 0$ *converges if* $p > 1$ *and diverges if* $0 < p \leq 1$.

The above series is called a p - series.
When $p = 1$ we get the series $\sum_{k=1}^{\infty} \frac{1}{k}$ which is called the harmonic series and which is of course divergent.

Example 13 *Determine if the series* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ *converges.*
Now $\frac{1}{\sqrt[3]{k}} = \frac{1}{k^{\frac{1}{3}}}$ *This means that the series is a p-series with* $p = \frac{1}{3}$. *From the last theorem we know that a p - series converges if* $p > 1$ *and diverges if* $0 < p \leq 1$. *Therefore the given series diverges.*

Theorem 6 (Ratio Test) *Let* $\sum_{k=1}^{\infty} a_n$ *be a series with non-zero terms.And let* $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

1. *The series converges if* $\rho < 1$
2. *The series diverges if* $\rho > 1$
3. *The test is inconclusive if* $\rho = 1$

Example 14 *Determine if* $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *converges or diverges.*
$a_{n+1} = \frac{2^{n+1}}{(n+1)!}, a_n = \frac{2^n}{n!}$
$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \frac{2}{n+1}$
$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0$
Therefore the series converges by the ratio test.

Theorem 7 (Alternating Series Test) *An alternating series*
$a_1 - a_2 + a_3 - a_4 + ... + (-1)^{k+1}a_k + ...$
or $-a_1 + a_2 - a_3 + ... + (-1)^k a_k + ...,$ *all* $a_k > 0$
converges if the following conditions are met:

1. $a_1 \geq a_2 \geq a_3 \geq ... \geq a_k \geq ...$

2. $\lim\limits_{k\to\infty} a_k = 0$

Definition 4 (Power Series) *An infinite series of the form*
$\sum\limits_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$
is called a power series in x.
An infinite series of the form $\sum\limits_{n=0}^{\infty} a_n (x-c)^n = \ a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$
is called a power series centered at c.

Theorem 8 (Convergence of a Power Series) *For a power series centered at c only one of the following is true.*

1. *The series converges only at* $x = c$.

2. *The series converges for all* x.

3. *There exists a positive real number R such that the series converges for* $|x - c| < R$ *and diverges for* $|x - c| > R$

In the third case the series converges in the interval $(c - R, C + R)$ *and diverges in intervals* $(-\infty, c - R)$ *and* $(c + R, \infty)$. *We would still need to check the endpoints* $c - R$ *and* $c + R$ *for convergence. The interval in which the series converges is called the interval of convergence.*

Definition 5 (Radius of Convergence) *The radius of convergence of a power series centered at c is*
$R = \lim\limits_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|, \ \ 0 \leq R \leq \infty.$

Example 15 *Find the radius of convergence of* $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$.

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1/n!}{1/(n+1)!} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty}(n+1) = \infty$$

A radius of convergence of infinity means that the power series converges for all real values of x.

Example 16 *Find the radius of convergence of* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$.

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n/2^n}{(-1)^{n+1}/2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\right| = \lim_{n\to\infty} 2 = 2$$

Since the center of the series is $c = -1$, we conclude that the series converges in the interval $(-1-2, -1+2) = (-3, 1)$. In fact if we check for convergence at the endpoints we find that the series diverges at the endpoints and $(-3, 1)$ is in fact the interval of convergence.

We now look at an important type of power series called the Taylor series. Here we'll show how to use derivatives of a function to write the power series for that function.

Definition 6 (Taylor Series) *If $f(x)$ has derivatives of all orders at c, then the power series for $f(x)$ centered at c is called the Taylor series for $f(x)$ centered at c and is given by*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + ...$$

If $c = 0$ then the Taylor series is called a Maclaurin series.

Example 17 *Find the Maclaurin series for $f(x) = e^x$.*

Now $f(0) = e^0 = 1$ and since $f'(x) = e^x$ and all higher derivatives of f also equal e^x. This implies that $f^{(n)}(0) = 1$ for all n.

Now by the definition of the Maclaurin series,

$e^x = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ...$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

$= \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Example 18 *Find the Taylor series for $f(x) = 1/x$, centered at 1.*

$f(x) = x^{-1} \Rightarrow f(1) = 1$

$f'(x) = -x^{-2} \Rightarrow f'(1) = -1$

$f''(x) = 2x^{-3} \Rightarrow f''(1) = 2$

$f'''(x) = -6x^{-4} \Rightarrow f'''(1) = -6$
$f^{(4)}(x) = 24x^{-5} \Rightarrow f^{(4)}(1) = 24$
$\Rightarrow f(x) = \frac{1}{x} = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!} + ...$
$= 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \frac{24(x-1)^4}{4!} - ...$
$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - ...$
$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$

Which is the Taylor series we wanted.