Lecture I - Indefinite Integration

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March 12, 2010

Today we will discuss Antiderivatives and Indefinite Integration. I am hoping that by the end of the lecture we should all be able to:

- reverse the process of differentiation to obtain an indefinite integral for simple functions.
- understand the role of the arbitrary constant.
- use a table of indefinite integrals of simple functions.
- understand and use the notation for indefinite integrals.
- use the constant multiple rule and the sum rule.

Ok let's get down to business. If you have done differentiation before then you have already dealt with antiderivatives. Let's look at an example.

Example 1 *Suppose we are given functions f and g such that their derivatives are $2x$ and $\cos x$ respectively. Can you guess what the functions are? Ok let's look closer, if $f(x) = x^2$ then $\frac{d}{dx}(x^2) = 2x$ and if $g(x) = \sin x$ then $\frac{d}{dx}(\sin x) = \cos x$. So $f = x^2$ and $g = \sin x$ seem to be the functions we want. However, $f = x^2 + 2$ and $g = \sin x + 5$ are also valid. In fact, $f(x) = x^2 + C$ and $g(x) = \sin x + C$, where C is an arbitrary constant, satisfy the requirement.*

The operation of determining the original function from its derivative is called antidifferentiation or integration.

Definition 1 *A function F is an antiderivative of a function f if for every x in the domain of f, $F'(x) = f(x)$.*

Notation

$$\int f(x)dx = F(x) + C$$

The "$\int$" is called the integral sign. $f(x)$ is called the integrand, "dx" is called the differential and "$F(x) + C$" is called the antiderivative of f or the indefinite integral of f.

Differentiation is the inverse of Integration

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

Integration is the inverse of Differentiation

$$\int f'(x)dx = f(x) + C$$

Basic Integration Rules

1. $\int kdx = kx + C$, k constant (Constant Rule)
2. $\int kf(x)dx = k\int f(x)dx$ (Constant Multiple Rule)
3. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$ (Sum Rule)
4. $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$(Difference Rule)
5. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$ (Simple Power Rule)

Let's take a look at some common integrals.

1. $\int \sin x dx = -\cos x + C$
2. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, where $n \neq -1$
   since $d(x^{n+1}) = (n+1)x^n dx$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int \sec^2 x dx = \tan x + C$
5. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
6. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

FACT: Antiderivatives are unique up to a constant.

Theorem 1 *If $F' = G'$, then $F(x) = G(x) + C$*

Well let's prove this one.
Proof: If $F' = G'$ then

$$(F - G)' = F' - G' = 0$$
$$\Rightarrow F(x) - G(x) = C$$
$$\Rightarrow F(x) = G(x) + C$$

EXAMPLES

Example 2 *Find $\int x^3(x^4+2)dx$*
Well we can multiply out the integrand to get $x^7 + 2x^3$.Hence,

$$\int x^3(x^4+2)dx = \int (x^7 + 2x^3)dx = \frac{x^8}{8} + \frac{2x^4}{4} + C$$
$$= \frac{x^8}{8} + \frac{x^4}{2} + C$$

Example 3 *Find $\int x^3(x^4+2)^5 dx$*
We can multiply out the integrand but that would be cumbersome.
We will use substitution which is a method we will discuss later on.
Let $u = x^4 + 2$ then $du = 4x^3dx$
So,

$$\int x^3(x^4+2)dx = \int (x^4+2)^5 x^3 dx = \int u^5 \frac{du}{4}$$
$$= \frac{u^6}{24} + C = \frac{(x^4+2)^6}{24} + C$$

Example 4 $\int e^{6x}dx = \frac{1}{6}e^{6x} + C$

Example 5 *Find $\int \frac{xdx}{\sqrt{1+x^2}}$*
Put $u = 1 + x^2$, $du = 2xdx$

$$\int \frac{xdx}{\sqrt{1+x^2}} = \int \frac{u^{-\frac{1}{2}}du}{2} = \ u^{\frac{1}{2}} + C = \sqrt{1+x^2} + C$$

Example 6 $\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C$

Example 7 $\int \frac{dx}{x\ln x} = \int \frac{d\ln x}{\ln x} = \ln(\ln x) + C$
If you have a problem with my $d\ln x$ *, just make your regular substitution* $u = \ln x$ *to get the desired result.*

Well I need you to remember the constant multiple rule and the sum rule so here goes ...

Constant Multiple Rule

$$\int mf(x)dx = m\int f(x)dx + C$$

Example 8 $\int 6\cos\theta d\theta = 6\int\cos\theta d\theta = 6\sin\theta + C$

Sum Rule

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx + C$$

Example 9 $\int (x+2)dx = \frac{x^2}{2}` + 2x + C$

Let's build a table of integrals:

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + C$
1	$x + C$
a	$ax + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln a} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$

Check p. 405 in Stroud (sixth edition) for more exercises to help you master the basics. In the next class we will finish up indefinite integrals and start looking at definite integrals.