A simple method for finding formulas for the sum of integer powers

Laurel Benn
Faculty of Technology
University of Guyana
Turkeyen, Guyana

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Abstract

Beginning science and engineering students often have difficulty when faced with the task of finding a closed form formula for something of the form $\sum_{i=1}^{n} i^k$ where k is a positive integer. In this paper a simple method for finding such closed formulas will be presented.

1 Introduction

Our goal is to find a closed form formula for $\sum_{i=1}^{n} i^k$ in general. We will begin the discussion by first finding formulas for a few lower order powers: $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$, $\sum_{i=1}^{n} i^3$, $\sum_{i=1}^{n} i^4$. After that , a general formula for the summation of i to any positive integral power k will be developed. To achieve our goal we make use of the binomial expansion.

2 Method

Suppose we need a closed form formula for $\sum_{i=1}^{n} i$. We need only start with

$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} (i-1)^2 + n^2$ from which we have

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 + n^2$$

$$\Rightarrow 2\sum_{i=1}^{n} i = n^2 + n$$

and so $\sum_{i=1}^{n} i = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$ as required.

So for this method we start with a power that is one more than the one we need. We then use the binomial expansion, cancel the higher power then solve for the power that we really need.

3 Closed form formula for $\sum\limits_{i=1}^{n} i^2$

Using the same method as before we write

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n}(i-1)^3 + n^3$$

$$\Rightarrow \sum_{i=1}^{n} i^3 = \sum_{i=1}^{n}(i^3 - 3i^2 + 3i - 1) + n^3$$

$$\Rightarrow 0 = -3\sum_{i=1}^{n} i^2 + \frac{n(3n+1)}{2} + n^3$$

$$\text{so } \sum_{i=1}^{n} i^2 = \left(n^3 + \frac{n(3n+1)}{2}\right)/3 = \frac{n(n+1)(2n+1)}{6}$$

4 Closed form formula for $\sum\limits_{i=1}^{n} i^3$

We start with $\sum\limits_{i=1}^{n} i^4 = \sum\limits_{i=1}^{n}(i-1)^4 + n^4$

$$\Rightarrow \sum_{i=1}^{n} i^4 = \sum_{i=1}^{n}(i^4 - 4i^3 + 6i^2 - 4i + 1) + n^4$$

$$\Rightarrow 0 = -4\sum_{i=1}^{n} i^3 + n(n+1)(2n+1) - n(2n+1) + n^4$$

$$\Rightarrow \sum_{i=1}^{n} i^3 = \frac{n^4 + n(n+1)(2n+1) - n(2n+1)}{4}$$

$$\Rightarrow \sum_{i=1}^{n} i^3 = \frac{n^2}{4}(n^2 + 2n + 1) = \frac{(n(n+1))^2}{4}$$

5 Closed form formula for $\sum\limits_{i=1}^{n} i^4$

We start with $\sum\limits_{i=1}^{n} i^5 = \sum\limits_{i=1}^{n}(i^5 - 5i^4 + 10i^3 - 10i^2 + 5i - 1) + n^5$

$$\Rightarrow 0 = -5\sum_{i=1}^{n} i^4 + 10\frac{(n(n+1))^2}{4} - 10\frac{n(n+1)(2n+1)}{6} + \frac{n(5n+3)}{2} + n^5$$

$$\Rightarrow \sum_{i=1}^{n} i^4 = \frac{n^5 + 10\frac{(n(n+1))^2}{4} - 10\frac{n(n+1)(2n+1)}{6} + \frac{n(5n+3)}{2}}{5}$$

$$\Rightarrow \sum_{i=1}^{n} i^4 = \frac{12n^5 + 30(n(n+1))^2 - \ 20n(n+1)(2n+1) + 6n(5n+3)}{60}$$
$$\Rightarrow \sum_{i=1}^{n} i^4 = \frac{6n^5 + 15(n(n+1))^2 - \ 10n(n+1)(2n+1) + 3n(5n+3)}{30}$$
$$\Rightarrow \sum_{i=1}^{n} i^4 = \frac{6n^5 + 15(n^4 + 2n^3 + n^2) - \ 10(2n^3 + 3n^2 + n) + 15n^2 + 9n}{30}$$
$$\Rightarrow \sum_{i=1}^{n} i^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

6 General formula for $\sum_{i=1}^{n} i^k$

We start with $\sum_{i=1}^{n} i^{k+1} = \sum_{i=1}^{n} (i-1)^{k+1} + n^{k+1}$

$$\Rightarrow 0 = \sum_{i=1}^{n} \sum_{j=1}^{k+1} (-1)^j \binom{k+1}{j} \ i^{k+1-j} + n^{k+1}$$
$$\Rightarrow (k+1) \sum_{i=1}^{n} i^k = n^{k+1} + \sum_{i=1}^{n} \ \sum_{j=2}^{k+1} (-1)^j \binom{k+1}{j} i^{k+1-j}$$
$$\Rightarrow \sum_{i=1}^{n} i^k = \frac{n^{k+1} + \sum_{i=1}^{n} \ \sum_{j=2}^{k+1} (-1)^j \binom{k+1}{j} i^{k+1-j}}{k+1}$$

Ok let's use the general form to find $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{n} \ i^2$

$$\sum_{i=1}^{n} i = \frac{n^2 + \sum_{i=1}^{n} 1}{1+1} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n^3 + \sum_{i=1}^{n} (3i-1)}{2+1} = \frac{n^3 + (\frac{3n+1}{2})n}{3} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

7 Summary

A simple method for the evaluation of $\sum_{i=1}^{n} i^k$ was developed. First we looked at formulas for $k = 1$ to 4. Then a general formula for these summations was developed. The binomial expansion was used extensively throughout to aid in the development of these formulas. The general formula obtained can also be easily coded using any programming language, and that can be the topic of another paper.