Test II solutions

EMT 121

July 23, 2009

Check back later in the week for the solutions to the series problems.

1. (a) $\int_{\sqrt{2}}^{15} 18x\,dx = \left[9x^2\right]_{\sqrt{2}}^{15} = 9[225-2] = 2007$

   (b) $\int x\sqrt{3x^2+2}\,dx = \frac{1}{6}\int \sqrt{3x^2+2}\,d(3x^2+2) = \frac{1}{9}(3x^2+2)^{\frac{3}{2}} + c$

   (c) $g(x) = \int_0^x \arctan(3t)\,dt$
   $\Rightarrow g'(x) = \arctan(3x)$ by the 2^{nd} fundamental Theorem of calculus.
   $\Rightarrow g''(x) = \frac{1}{(3x)^2+1}\cdot 3 = \frac{3}{9x^2+1}$
   $\Rightarrow g'(\frac{1}{3}) = \arctan(3\cdot\frac{1}{3}) = \arctan(1) = \frac{\pi}{4}$
   $\Rightarrow g''(\frac{1}{3}) = \frac{3}{9(\frac{1}{3})^2+1} = \frac{3}{2}$.

2. (a) $A = \int_0^2 (2x-x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$

   (b) $V = \pi\int_0^4 \left(y - \frac{y^2}{4}\right)dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \pi\left[8 - \frac{16}{3}\right] = \frac{8\pi}{3}$

3. (a) $\int_0^1 x\cos(ax)\,dx = \left[\frac{x\sin(ax)}{a} + \frac{\cos(ax)}{a^2}\right]_0^1 = \frac{\sin a}{a} + \frac{\cos a}{a^2} - \frac{1}{a^2}$
   $= \frac{a\sin a + \cos a - 1}{a^2}$

   (b) $\int \frac{30}{x^2-25x+100}\,dx = \int \frac{-2}{x-5}\,dx + \int \frac{2}{x-20}\,dx$
   $= -2\ln|x-5| + 2\ln|x-20| + c$

   (c) $\int \sec^3\theta\tan\theta\,d\theta = \int \sec^2\theta\,d\sec\theta = \frac{\sec^3\theta}{3} + c$

   (d) $\int \frac{x-1}{x(x+1)^2}\,dx = \int \frac{-1}{x}\,dx + \int \frac{1}{x+1}\,dx + \frac{2}{(x+1)^2}\,dx$
   $= -\ln|x| + \ln|x+1| - \frac{2}{x+1} + c$

   (e) $\int \frac{2x-1}{2x^2-2x+3}\,dx = \frac{1}{2}\ln|2x^2-2x+3| + c$

   (f) $\int \sqrt{e^3x}\,dx = e^{\frac{3}{2}}\int x^{\frac{1}{2}}\,dx = e^{\frac{3}{2}}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2\sqrt{e^3}x^{\frac{3}{2}}}{3} + c$

(g) $\displaystyle\int \frac{x^{\frac{2}{3}}}{x+1}\,dx$
Well I feel we did a good job solving this one in class.

4. Determine whether the following integrals converge or diverge.If the integral converges, evaluate it.

(a) $\displaystyle\int_0^\infty e^{-2x}\,dx = \lim_{M\to\infty}\int_0^M e^{-2x}\,dx \ = \lim_{M\to\infty}\left[-\frac{e^{-2x}}{2}\right]_0^M$
$\displaystyle= \lim_{M\to\infty}\left[\frac{1}{2}-\frac{e^{-2M}}{2}\right] = \ \frac{1}{2}$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \sec t\tan t\,dt$

(c) $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

We'll take the class solutions for (b) and (c).

5. Given the function f at the following values:

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

Approximate $\displaystyle\int_{1.8}^{2.6} f(x)\,dx$ using

(a) the Trapezoidal rule
$\displaystyle\int_{1.8}^{2.6} f(x)\,dx \approx \frac{(2.6-1.8)}{8}\,[3.12014+2(4.42569+6.04241+8.03014)+10.46675] = 5.058337$

(b) Simpson's rule
$\displaystyle\int_{1.8}^{2.6} f(x)\,dx \approx \frac{(2.6-1.8)}{12}\,[3.12014+4(4.42569+8.03014)+2(6.04241)+10.46675] = 5.033002$

6. Use the Trapezium Rule with $n=4$ to approximate $\displaystyle\int_1^2 x\ln x\,dx$
First set up a table of values as follows:

x	1	1.25	1.5	1.75	2.0
$f(x)$	0	0.2789	0.6082	0.9793	1.3863

This implies that,
$\displaystyle\int_1^2 x\ln x\,dx \approx \ \frac{1}{8}[0+2(0.2789+0.6082+0.9793)+1.3863] = 0.6398875$

7. For each of the following series , determine whether it converges or diverges.

(a) $\sum_{n=4}^{\infty} \frac{1}{n}$

This series diverges since it is a known divergent series with the first three terms taken out. We could also use the Integral or Ratio test to show that this series diverges.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

Now,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = \sum_{n=1}^{\infty} \frac{-1/2}{n+3} + \sum_{n=1}^{\infty} \frac{1/2}{n+1} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} + ... - \frac{1}{n+2} + \frac{1}{n+1} - \frac{1}{n+3} + ...\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}.$$

Series converges with sum $\frac{5}{12}$.

(c) $\sum_{n=1}^{\infty} \frac{1}{n!}2^n$

Here I would use the ratio test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right|$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0$

This means that the series converges.

(d) $\sum_{n=1}^{\infty} \frac{n^2}{n^2+10000}$

For this one since $\lim_{n\to\infty} \frac{n^2}{n^2+10000} = 1 \Rightarrow$ The series diverges by the divergence test.

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

but $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$. Therefore $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

(f) $\sum_{n=1}^{\infty} \frac{1}{n+6} = \sum_{n=7}^{\infty} \frac{1}{n}$

which is a known divergent series with the first six terms taken out. Therefore $\sum_{n=1}^{\infty} \frac{1}{n+6}$ diverges.

(g) $\sum_{k=1}^{\infty} \frac{3}{5k}$

(h) $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$

(i) $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

8. Find the sum of

   (a) $\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{4^n}$

   (b) $\sum_{n=1}^{\infty} \frac{1}{5^n} - \frac{1}{n(n+1)}$

9. Determine the radius of convergence of the following Power Series.

$$\sum_{n=1}^{\infty} \frac{n}{6^n}(x-3)^n$$