EMT 121

Practice Final Exam

Friday June 4, 2010

1. For the series, $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+1)^n}{2^{n+1}}$, find

   (a) the radius of convergence.
   $R = \displaystyle\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n}{2^{n+1}} \cdot \frac{2^{n+2}}{n+1} \right| = 2$

   (b) the interval of convergence.
   $(-3, 1)$

2. Express e^x as a Maclaurin series.

   $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3. Determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 - 3}$ converges or diverges.

   Using the limit comparison test (comparing with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$)we get,

   $\displaystyle\lim_{n\to\infty} \frac{k^2}{k^2 - 3} = 1$. Since this limit is between 0 and ∞ it tells us that since $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2}$ converges then $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 - 3}$ also converges.

4. Determine if $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent or divergent.

$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} (1 - \frac{1}{b}) = 1$

$\Rightarrow \int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

5. Find the sum of $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

$= \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... + \frac{1}{k} - \frac{1}{k+1} + ...$

$\Rightarrow S_k = 1 - \frac{1}{k+1} \Rightarrow \lim_{k\to\infty} S_k = 1$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

6. Evaluate $\int \frac{\ln(x+5)}{x+5}\,dx$

$= \int \ln(x+5)d(\ln(x+5)) = \frac{(\ln(x+5))^2}{2} + C$

7. Evaluate $\int_0^{\pi} x \sin x\,dx$

$= [-x\cos x + \sin x]_0^{\pi} = -\pi\cos\pi + \sin\pi = -\pi\cdot(-1) + 0 = \pi$

8. Evaluate $\int \tan^3 x \sec^2 x \, dx$

$$= \int \tan^3 x \, d \tan x = \frac{\tan^4 x}{4} + C$$

9. Evaluate $\int \frac{dx}{\sqrt{16 - x^2}}$

Let $x = 4 \sin \theta \Rightarrow dx = 4 \cos \theta d\theta$

$$\Rightarrow \int \frac{dx}{\sqrt{16 - x^2}} = \int \frac{4 \cos \theta}{4 \cos \theta} d\theta = \int d\theta = \theta + C = \sin^{-1}(x/4) + C$$

10. Estimate the value of $\int_0^2 e^{x^2} \, dx$ with $n = 2$, using the Trapezoidal rule.

x	$f(x)$
0	1
1	e
2	e^4

$$\int_0^2 e^{x^2} \approx \frac{1}{2}(1 + e^4 + 2e) \approx 30.5$$

11. Consider the region R, in the first quadrant, bounded above by $y = x$ and below by $y = x^2$.

    (a) Find the area of R.
    $A = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

    (b) Find the volume of the solid that is obtained by rotating R about the y-axis.
    $V = \int_0^1 2\pi x(x - x^2)\, dx = \ 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \ \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$
    $= 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) \ = \frac{\pi}{6}$

12. Solve the following system of equations using Gauss-Jordan elimination.
    $2x_2 - 2x_3 = -8$
    $x_1 + x_2 + x_3 = 2$
    $x_1 + 2x_2 = -2$

$$\begin{pmatrix} 0 & 2 & -2 & -8 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & -8 \\ 1 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1 + R_3} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & -8 \\ 0 & 1 & -1 & -4 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & -1 & -4 \end{pmatrix} \xrightarrow{-R_2 + R_1} \begin{pmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & -1 & -4 \end{pmatrix} \xrightarrow{-R_2 + R_3} \begin{pmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the last row of the last matrix consists of only zeros we stop the Gauss-Jordan elimination process and solve the remaining 2 equations. We get $x_2 - x_3 = -4 \Rightarrow x_2 = x_3 - 4$, $x_1 + 2x_3 = 6 \Rightarrow x_1 = 6 - 2x_3$.
If we let $t = x_3$ then the solution to this system becomes $x_1 = 6 - 2t$, $x_2 = t - 4, x_3 = t$. Where t is any real number. For example, if we let $t = 0$, then $x_1 = 6$, $x_2 = -4$, $x_3 = 0$ is a solution to this system.However there are infinitely many solutions since t can be any real number.