University of Guyana
Faculty of Technology

EMT 111 - Final Examination - Solutions

December 20, 2010

Directions: Answer all questions. Show all work. Calculators ARE allowed.

1. Evaluate.(9 points)

   (a) $\log_{64} 4$. Since $4^3 = 64 \Rightarrow 64^{\frac{1}{3}} = 4 \Rightarrow \log_{64} 4 = \boxed{\frac{1}{3}}$

   (b) $\log \sqrt{10}$. Here $\log \sqrt{10} = \log 10^{1/2} = \frac{1}{2} \log 10 = \boxed{\frac{1}{2}}$

   (c) $(2x^3 + x - 1) \div (x^2 + 1)$. Using long division we get $2x$ with $-x - 1$ as a remainder.

2. Solve for x.(9 points)

   (a) $\log(x+2) + \log(x-1) = 1 \Rightarrow (x+2)(x-1) = 10$
   $\Rightarrow x^2 + x - 12 = 0 \Rightarrow (x+4)(x-3) = 0 \Rightarrow \boxed{x = 3}$

   (b) $x^2 - x \leq 6 \Rightarrow x^2 - x - 6 \leq 0 \Rightarrow (x-3)(x+2) \leq 0$
   $\Rightarrow \boxed{x \in [-2, 3]}$

   (c) $10^{x+4} = 100 \Rightarrow 10^{x+4} = 10^2 \Rightarrow x + 4 = 2$
   $\Rightarrow \boxed{x = -2}$

3. Compute the derivative of the following functions.(20 points)

   (a) $s(t) = \frac{1}{t} - \frac{1}{t^2} - \frac{1}{\sqrt{t}} \Rightarrow \boxed{s'(t) = -\frac{1}{t^2} + \frac{2}{t^3} - \frac{1}{2\sqrt{t^3}}}$

   (b) $h(x) = \frac{x^5 - 4x^2}{x^3} = x^2 - \frac{4}{x} \Rightarrow \boxed{h'(x) = 2x + \frac{4}{x^2}}$

   (c) $f(x) = e^{\sec x} \Rightarrow \boxed{f'(x) = e^{\sec x} \sec x \tan x}$

   (d) $g(x) = \ln(\cos x) \Rightarrow g'(x) = -\frac{\sin x}{\cos x} = \boxed{-\tan x}$

   (e) $s(t) = \sin(\sqrt{1-t^2}) \Rightarrow s'(t) = \cos(\sqrt{1-t^2}) \underbrace{(\sqrt{1-t^2})'}_{\frac{-2t}{2\sqrt{1-t^2}}}$

   $\Rightarrow \boxed{s'(t) = \frac{-t\cos(\sqrt{1-t^2})}{\sqrt{1-t^2}}}$

4. Evaluate the following integrals. (24 points)

(a) $\int x\sqrt{x^2+2}\,dx = \frac{1}{2}\int \sqrt{x^2+2}\,d(x^2+2) \;= \frac{1}{2}(x^2+2)^{3/2}\cdot\frac{2}{3}+C$
$= \boxed{\frac{1}{3}(x^2+2)^{3/2}+C}$

(b) $\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^{x^2}\,d(x^2) \;= \frac{1}{2}(e^1-e^0) = \boxed{\frac{1}{2}(e-1)}$

(c) $\int \frac{\cos(1-x)}{3}\,dx = \boxed{-\frac{\sin(1-x)}{3}+C}$

(d) $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$
Let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta$
$\Rightarrow \int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{8\sin^3\theta\cdot 2\cos\theta d\theta}{2\cos\theta}$
$= 8\int \sin^3\theta d\theta = 8\int (1-\cos^2\theta)\sin\theta d\theta$
$= 8[-\cos\theta + \frac{\cos^3\theta}{3}] + C = \boxed{8[-\frac{\sqrt{4-x^2}}{2} + \frac{(4-x^2)^{3/2}}{24}] + C}$

(e) $\int \frac{5x-2}{(x-1)(x-2)}\,dx = \int \frac{-3dx}{x-1} + \int \frac{8dx}{x-2} = \boxed{-3\ln|x-1| + 8\ln|x-2| + C}$

(f) $\int_0^{\frac{\pi}{4}} \cos^3 x\sin x\,dx = \; -\int_0^{\frac{\pi}{4}} \cos^3 x\,d(\cos x) = \left|-\frac{\cos^4 x}{4}\right|_0^{\frac{\pi}{4}}$
$= \frac{-(\frac{1}{\sqrt{2}})^4}{4} + \frac{1}{4} = \frac{-1}{16} + \frac{1}{4} = \boxed{\frac{3}{16}}$

5. Use the fact that 12 fluid ounces is approximately 6.89π cubic inches to find the dimensions of the 12-ounce can that can be constructed using the least amount of metal. (5 points)
$V = \pi r^2 h = 6.89\pi \Rightarrow h = \frac{6.89}{r^2}$
$SA = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\frac{6.89}{r^2} \;= 2\pi[r^2 + \frac{6.89}{r}]$
Need r that would minimize the surface area, SA.
$SA'(r) = 2\pi[2r - \frac{6.89}{r^2}] = 2\pi[\frac{2r^3-6.89}{r^2}]$
Setting $SA'(r) = 0$ we have $2r^3 = 6.89$ and $r = \sqrt[3]{3.445} = \boxed{1.51}$
$\Rightarrow h = \frac{6.89}{1.51^2} = \boxed{3.02}$
Therefore the can must have a radius of 1.51 inches and a height of 3.02 inches.

6. Find the absolute minimum and maximum values of the function $f(x) = \frac{1}{3}x^3 - 9x + 2$ on the interval $[0, 2]$. (5 points)
$f'(x) = x^2 - 9 = (x-3)(x+3) = 0 \Rightarrow x = 3, x = -3$ are critical numbers.
However, $x = 3$ and $x = -3$ are outside the interval $[0, 2]$.
Therefore we need not evaluate the function at these points.

Now $f(0) = \boxed{2} \leftarrow$ absolute maximum
and $f(2) = \frac{8}{3} - 18 + 2 = \boxed{-\frac{40}{3}} \leftarrow$ absolute minimum

7. (a) Prove: $2\tan x \sec x = \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$ (5 points)
$RHS = \frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{1+\sin x-(1-\sin x)}{1-\sin^2 x}$
$= \frac{2\sin x}{\cos^2 x} = \frac{2\sin x}{\cos x} \cdot \frac{1}{\cos x} = 2\tan x \sec x = LHS$

(b) An ant starts at the point (1,0) on the unit circle and walks counterclockwise 4 units around the circle. Find the coordinates of the final location of the ant. (5 points)
Coordinates are $(x, y) = (r\cos\theta, r\sin\theta)$
but $r = 1$ (unit circle) and $\theta = l = 4$
Therefore coordinates of the ant is $(\cos 4, \sin 4)$ where the angles are in radians.

(c) Show that the triangle with sides 3,4 and 5 is a right-angled triangle. (3 points)
If 3,4,5 represent the sides of a right triangle then 5^2 must equal $3^2 + 4^2$ (By the Pythagorean Theorem).
Let's check:
$5^2 \stackrel{?}{=} 3^2 + 4^2$
$25 \stackrel{?}{=} 9 + 16$
$25 \stackrel{?}{=} 25$ yes!
This implies that a triangle with sides 3, 4, 5 is a right triangle.

8. When a resistor of R ohms is connected across a battery with electromotive force E volts and internal resistance r ohms, a current of I amperes will flow, generating P watts of power,
where $I = \frac{E}{r+R}$ and $P = I^2R$.
Assuming r is a constant, what choice of R results in maximum power? (5 points)
$P = I^2R = \left(\frac{E}{r+R}\right)^2 R = \frac{E^2R}{(r+R)^2}$
$P'(R) = \frac{(r+R)^2E^2-2E^2R(r+R)}{(r+R)^4} = 0$
$\Rightarrow (r+R)E^2[r+R-2R] = 0 \Rightarrow r - R = 0 \Rightarrow \boxed{R = r}$ gives maximum power.

9. A fish pond is nearly circular, is 20 feet deep at its center ,and has a radius of 200 feet. The bottom of the pond can be modeled by

$$y = 20[(0.005x)^2 - 1]$$

How much water can this pond hold? (5 points)
Using the method of shells, we have
$V = \int_0^{200} 2\pi x(0 - 20[(0.005x)^2 - 1])\, dx$

$$= 2\pi \int_0^{200} 20x[1-(0.005x)^2]\,dx$$
$$= 40\pi \int_0^{200} (x - 25\times 10^{-6}x^3)\,dx$$
$$= 40\pi \left[\frac{x^2}{2} - 25\times 10^{-6}\times\frac{x^4}{4}\right]_0^{200} = 40\pi\left(\frac{40000}{2} - \frac{25\times 10^{-6}\times 16\times 10^8}{4}\right)$$
$$= 40\pi[20000-10000] = 400000\pi = \boxed{\text{1,256,637 cubic ft.}}$$

10. John is a pilot with TransGuyana Airways. Assume that during one storm he flies with the wind (a tailwind) at 300 miles per hour. Leaving the storm, he flies against the wind(a headwind) at 225 miles per hour. If he did not adjust his plane's speed, find the speed of the plane in still air and the speed of the wind. (5 points)
Let x = speed of plane in still air, and y = speed of the wind then

$$x + y = 300$$
$$x - y = 225$$

$\Rightarrow 2x = 525 \Rightarrow \boxed{x = 262.5}$ mph
$\Rightarrow 2y = 75 \Rightarrow \boxed{y = 37.5}$ mph